Multisolitons, or the discrete eigenfunctions of the recursion operator of nonlinear evolution equations: II. Background

1987 J. Phys. A: Math. Gen. 20 375

(http://iopscience.iop.org/0305-4470/20/2/023)

J. Phys. A: Math. Gen. 20 (1987) 375-388. Printed in the UK

Multisolitons, or the discrete eigenfunctions of the recursion operator of non-linear evolution equations: II. Background

Benno Fuchssteiner and Raju N Aiyer†
University of Paderborn, D 4790 Paderborn, West Germany

Received 19 February 1986

Abstract. Different definitions of the notion of multisoliton are investigated. It is demonstrated that, even in more complicated cases (like the CDGSK equation), multisolitons can be completely characterised in terms of the discrete spectrum of the strong symmetry and that degeneracy of this spectrum leads to resonance solitons (yielding singular solitons in case of the KdV equation and non-singular ones in other cases). Furthermore a method is described which allows us to compute strong symmetries out of auto-Bäcklund transformations for non-linear systems.

1. Introduction

In [1] we explicitly computed the general solution of the CDGSK (Caudrey-Dodd-Gibbon-Sawada-Kotera) equation [2, 3] obtained by iterating the auto-Bäcklund transformation once [3, 4]. The explicit computation has shown that this general soliton solution does correspond to spectral decomposition of the recursion operator of the CDGSK, as was stated in [5]. But apart from this expected behaviour these multisolitons had some novel features with respect to their time evolutions. As a consequence this led to an unexpected form of the corresponding linear dependence of infinitesimal generators of one-parameter symmetry groups determined by these multisoliton solutions. For the popular completely integrable systems, the corresponding solution is characterised by a linear dependence between the generators of x and time translation, which is not the case for the CDGSK. Another surprising consequence was that, with respect to time evolution, even the one-soliton behaved differently from what was expected. To be precise: apart from what usually [6] is considered to be a one-soliton we found a class of solutions of the corresponding equation (kernel equations (26)) being equal to a one-soliton only for a special choice of the integrating parameters but leading nevertheless, as in the case of one-solitons, to a discrete one-point spectrum of the recursion operator.

In order to clear up the confusion which may come out of these surprising discoveries, in this paper we study different notions of 'multisoliton solutions' and compare these with the spectral properties of the recursion operator. One discovery resulting out of this comparison is that auto-Bäcklund transformations in general determine isospectral problems for the flow under consideration and that these isospectral problems are equivalent to those given by the recursion operators. In this context it then turns out that, in the case of the CDGSK, the novel features of the multisolitons

† Permanent address: Laser Division, Bhabha Atomic Research Centre, Bombay 400085 India.

are due to spectral degeneracy or due—if one wishes to adopt this viewpoint—to non-linear resonance. The fact that the corresponding 'resonance solitons' do not occur in the well known cases (like KdV, etc) is mostly due to the special solution-manifold one usually studies for these equations.

Let us begin with the following.

Situation 1. On some infinite-dimensional manifold of suitable functions in the real variable x we consider an equation

$$u_t = K(u) \qquad u = u(x, t) \tag{1}$$

such that there is a hereditary operator [7, 8] $\Phi(u)$ generating the vector field $K(u)$ out of the generator of the translation group, i.e.

$$K(u) = \Phi(u)^M u_x. \tag{2}$$

The property of hereditariness then implies that the vector fields

$$K_n(u) = \Phi(u)^n u_x \qquad n = 0, 1, \ldots. \tag{3}$$

do commute in the Lie algebra of vector fields. Because of $K(u) = K_M(u)$ we have then constructed infinitely many generators of one-parameter symmetry groups for (1).

Furthermore, we assume that an auto-Bäcklund relation (ABT) for (1) is given by

$$B(u, \bar{u}, \lambda) = 0. \tag{4}$$

This means that $B(\ldots, \lambda)$ is a one-parameter family of C^∞ functions on the product of the manifold under consideration such that for each λ the submanifold determined by (4) is invariant under (1), i.e.

$$B_u[K(u)] + B_{\bar{u}}[K(\bar{u})] = 0 \tag{5}$$

when $B(u, \bar{u}, \lambda) = 0$. Here

$$B_u[K(u)] = (\partial/\partial\varepsilon|_{\varepsilon=0}) B(u + \varepsilon K(u), \bar{u}, \lambda) \tag{6}$$

and $B_{\bar{u}}[K(\bar{u})]$ denote the variational derivatives with respect to u (and $\bar{u}$) in the direction of $K(u)$ (and $K(\bar{u})$). Equation (5) is equivalent to $B_t(u, \bar{u}, \lambda) = 0$ whenever $u = u(t)$ and $\bar{u} = \bar{u}(t)$ are solutions of (1) related for one time t_0 by $B(u(t_0), \bar{u}(t_0), \lambda) = 0$.

Later on we show that the operator $\Phi(u)$ can be computed from (4). Since we are demonstrating this explicitly for two examples we may as well introduce these two examples at this point in order to illustrate situation 1.

Example 1. The following operator is hereditary [7]:

$$\Phi(u) = D^2 + 2u + 2DuD^{-1} \tag{7}$$

where D denotes the operator of taking the x derivative. The Korteweg-de Vries equation is of the form

$$u_t = \Phi(u) u_x = u_{xxx} + 6uu_x. \tag{8}$$

An auto-Bäcklund relation for this equation is well known [9]:

$$B(u, \bar{u}, \lambda) = (u + \bar{u}) + \tfrac{1}{2}(D^{-1}(u - \bar{u}))^2 - \bar{\lambda} = 0. \tag{9}$$

Example 2. The operator

$$\Phi(u) = D + DuD^{-1} \tag{10}$$

is hereditary [7] and

$$u_t = \Phi(u)u_x = u_{xx} + 2uu_x \tag{11}$$

yields the Burgers equation, for which an auto-Bäcklund relation is easily found (see [10]):

$$B(u, \bar{u}, \lambda) = \exp(-D^{-1}(u - \bar{u})) - cu + \lambda = 0 \tag{12}$$

where c is some arbitrary constant.

2. The different conditions for multisoliton solutions

A decisive role in characterising soliton solutions is played by the linear hull of the symmetry generators $K_n(u)$, $n = 0, 1, \ldots$. By $\boldsymbol{K}$ we denote the non-trivial (not all coefficients equal to zero) linear combinations of the $K_n(u)$, $n = 0, 1, \ldots$. If $L(u) = \Sigma\, a_n K_n(u)$ is an element of $\boldsymbol{K}$ then the polynomial $P_L(\xi) = \Sigma\, a_n \xi^n$ is called its characteristic polynomial. Observe that $L(u) = P_L(\Phi(u))u_x$. L is said to be non-degenerate if the zeros of its characteristic polynomial have only multiplicity 1. The different requirements on u for being a multisoliton solution can be summarised as follows.

(i) Some $L \in \boldsymbol{K}$ is equal to zero.

(ii) The generator of the translation group u_x can be decomposed into non-zero eigenvectors $\omega_0, \omega_1, \ldots, \omega_N$ of $\Phi(u)$ i.e.

$$u_x = \omega_0 + \omega_1 + \ldots + \omega_N.$$

(iii) Some $L \in \boldsymbol{K}$ can be decomposed into eigenvectors of $\Phi(u)$.

(iv) Every $L \in \boldsymbol{K}$ can be decomposed into eigenvectors of $\Phi(u)$.

(v) By iteration of the auto-Bäcklund transformation u can be obtained from the zero function, i.e. there are functions $u_0 = 0, u_1, u_2, \ldots, u_{N-1}$, $u_N = u$ and numbers $\lambda_1, \lambda_2, \ldots, \lambda_N$ such that

$$B(u_{n+1}, u_n, \lambda_{n+1}) = 0 \qquad n = 0, 1, \ldots, N-1. \tag{13}$$

(vi) u is an element of a finite-dimensional submanifold which is invariant with respect to all the flows:

$$u_t = K_n(u). \tag{14}$$

Since being a multisoliton solution should not change with the time evolution of (1) it is important that all the submanifolds determined by definitions (i)-(v) are invariant under (1). In other words, whenever the initial condition $u(t = 0)$ fulfils one of these conditions, then the solution $u(t)$ fulfils the same condition for all time t. For conditions (ii)-(iv) the proof of this fact can be found in [7], or better in [8]. In the case of (i) this follows trivially since the $K_n(u)$ are symmetry generators. In the case of (v) this is a direct consequence of (5) and in (vi) it is explicitly required in the definition.

Lemma 1. The following implications hold: condition (ii)$\Leftrightarrow$condition (iv)$\Rightarrow$condition (i)$\Leftrightarrow$condition (iii).

Before proving this lemma we mention that, in fact, the implication from (iv) to (ii) cannot be reversed (example 3) without adding a non-degeneracy condition (theorem 1).

Proof. The implications (iv)⇒(ii)⇒(iii) are obvious.

(iii)⇒(i). Assume that we have a decomposition

$$L=\sum a_m K_m=\omega_0+\ldots+\omega_N$$

into eigenvectors with correspondig eigenvalues $\lambda_0,\ldots,\lambda_N$. Consider the polynomial

$$P(\Phi)=(\Phi(u)-\lambda_0)(\Phi(u)-\lambda_1)\ldots(\Phi(u)-\lambda_N)=\sum b_n\Phi^n.$$

Obviously $P(\Phi)L=0$. Using $\Phi^n K_m=K_{m+n}$ we obtain

$$\sum_m\sum_n a_m b_n K_{n+m}=0.$$

(i)⇒(iii). Assume $L=\Sigma\, a_m K_m$. We consider polynomials $P(\Phi)$ with $P(\Phi)u_x=0$. There is at least one such polynomial, namely $\bar{P}(\Phi)=\Sigma\, a_m\Phi^m$, so there must be a minimal one, say

$$P_0(\Phi)=(\Phi-\lambda_0)(\Phi-\lambda_1)\ldots(\Phi-\lambda_N).$$

Consider

$$\Pi(\Phi)=(\Phi-\lambda_1)\ldots(\Phi-\lambda_N)=\sum\beta_n\Phi^n.$$

Then

$$\omega=\Pi(\Phi)u_x=\sum\beta_n K_n(u)$$

is non-zero because $P_0(\Phi)$ was minimal. We have $(\Phi-\lambda_0)\omega=0$, hence $\Sigma\,\beta_n K_n$ is an eigenvector of $\Phi(u)$ with eigenvalue λ_0.

(ii)⇒(iv). If u_x can be decomposed into eigenvectors of $\Phi(u)$

$$u_x=\omega_0+\omega_1+\ldots+\omega_N$$

with eigenvalues $\lambda_0,\ldots,\lambda_N$ then

$$K_m(u)=\Phi^m(u)u_x=\sum\lambda_n^m\omega_n.$$

Hence condition (iv) holds.

The proof of this simple lemma shows clearly that, under additional conditions concerning the multiplicity of characteristic polynomials, we can achieve the equivalence of conditions (i)-(iv).

If some $L\in\boldsymbol{K}$ can be written as sum of Φ eigenvectors

$$L=\omega_0+\omega_1+\ldots+\omega_N$$

with corresponding eigenvalues $\lambda_0,\ldots,\lambda_N$ then the union of $\{\lambda_0,\ldots,\lambda_N\}$ with the zeros of the characteristic polynomial of L are called the zero set of L. This name has been chosen because it turns out that the above condition on L is equivalent to condition (i), where the zeros of the characteristic polynomial are equal to this zero set.

Theorem 1. The following are equivalent.

(a) There is a non-degenerate $L = \Sigma\, a_m K_m$ such that $L(u) = 0$.

(b) The generator of the translation group can be decomposed

$$u_x = \omega_0 + \omega_1 + \ldots + \omega_N$$

into eigenvectors of $\Phi(u)$.

(c) Some non-degenerate $H = \Sigma\, b_n K_n$ can be decomposed

$$H = \bar{\omega}_0 + \bar{\omega}_1 + \ldots + \bar{\omega}_M$$

into a (possibly empty) set of eigenvectors of $\Phi(u)$ such that the set of the corresponding eigenvalues is disjoint from the zeros of the characteristic polynomial of H.

If one of these conditions is fulfilled then the zero sets of $L(u)$, $H(u)$ and u_x coincide.

Proof. We prove (a)⇒(c)⇒(b)⇒(a).

(a)⇒(c). Of course, (a) corresponds to a decomposition into an empty set of eigenvectors.

(c)⇒(b). Let $\lambda_0, \ldots, \lambda_N$ be the set consisting of the different eigenvalues (occurring in the definition of H) and the zeros of the characteristic polynomial of H. Then, obviously,

$$(\Phi - \lambda_0)(\Phi - \lambda_1) \ldots (\Phi - \lambda_N) u_x = 0.$$

Consider the polynomial $P(\xi) = (\xi - \lambda_0)(\xi - \lambda_1) \ldots (\xi - \lambda_N)$ and define the polynomials $\Pi_n(\xi)$, $n = 0, \ldots, N$ by $(\xi - \lambda_n)\Pi_n(\xi) = P(\xi)$. Since all zeros of $P(\xi)$ have multiplicity 1 we know from elementary calculus that

$$1 = \sum_{n=0}^{N} \alpha_n \Pi_n(\xi)$$

where the α_n are given by

$$\alpha_n = (P'(\xi)|_{\xi = \lambda_n})^{-1}.$$

Hence

$$u_x = \sum_{n=0}^{N} \alpha_n \Pi_n(\Phi) u_x. \tag{15}$$

Now, introducing $\omega_n = \Pi_n(\Phi) u_x$ we see from

$$(\Phi - \lambda_n)\omega_n = P(\Phi) u_x = 0$$

that ω_n must be an eigenvector of Φ with eigenvalue λ_n and that (15) is the desired decomposition.

(b)⇒(a). Let $\lambda_0, \ldots, \lambda_N$ be the set of different eigenvalues occurring in the decomposition of u_x. Then obviously

$$(\Phi - \lambda_0)(\Phi - \lambda_1) \ldots (\Phi - \lambda_N) u_x = 0.$$

Thus, if $P(\xi) = (\xi - \lambda_0)(\xi - \lambda_1) \ldots (\xi - \lambda_N) = \Sigma\, a_n \xi^n$, we have found some

$$L(u) = \sum a_n K_n(u)$$

with $L(u) = 0$ such that the zeros of the characteristic polynomial are given by the λ.

Henceforth, we call a solution u fulfilling any of the equivalent conditions in theorem 1 a multisoliton, whereas those solutions u fulfilling the more general condition (i) (or (iii)) without the non-degeneracy requirement will be called resonance multisolitons. Of course, any multisoliton is also a resonance multisoliton, since whenever it fulfils a condition with a non-degeneracy requirement, then it also fulfils a condition violating the non-degeneracy requirement. For example, when $K_1(u)-\lambda u_x=0$ then

$$K_2(u)-2\lambda K_1(u)+\lambda^2 u_x=(\Phi-\lambda)(K_1(u)-\lambda u_x)=0.$$

But there may be resonance multisolitons which are not multisolitons in the strict sense. Before we give examples for such a situation we would like to clarify the interrelation with condition (vi). The connection with condition (v) will be cleared up in the next section.

Remark 1. Condition (vi) implies that u is a resonance multisoliton. In case the $K_n(u)$, $n=1,2,\ldots$, are polynomials in derivatives of u, condition (vi) is equivalent to u being a resonance multisoliton.

Proof. Since the manifold under consideration is invariant with respect to the K_n, the vector fields K_n are in the tangent bundle of this manifold. Furthermore, at each point u, they must be linearly dependent since the tangent planes are finite dimensional. But such a linear dependence is preserved along the orbits of the flows $u_t=K_n(u)$ since the $K_m(u)$ are symmetry generators for all these flows. Hence condition (i) must hold. On the other hand, if condition (i) holds and if the $K_n(u)$ are of the required form, then condition (i) yields an ordinary differential equation whose solution manifold is finite dimensional.

Of course, in condition (vi) the requirement that the finite-dimensional manifold must be invariant against all the flows $u_t=K_n(u)$ is essential. Examples for which the less stringent condition of being invariant only against (1) is fulfilled are given by the similarity solutions.

Physicists like to define multisolitons by their asymptotic behaviour with respect to large time t. Of course, this viewpoint is more special than the one we have adopted here, but nevertheless it is compatible with ours. For example, start with the one-soliton given by the ordinary differential equation $(\Phi(u)-\lambda)u_x=K_1(u)-\lambda u_x=O$ which has the solution, $s_\lambda(x)$, say. Since $u_t-\lambda^M u_x=0$, then $s_\lambda(x+\lambda^M t)$ must be the corresponding solution of (1). Now, assume that for all λ these $s_\lambda(x)$ vanish suitably rapidly at $|x|\to\infty$. Then, if the $K_n(u)$ are local with respect to x, for example if all $K_n(u)$ are polynomials in u and its derivatives, then any solution u which decomposes for $|t|\to\infty$ into faraway one-solitons $s_{\lambda_1}, s_{\lambda_2},\ldots, s_{\lambda_N}$ (with distinct λ) must be a solution of $0=\Pi_{n=0}^{N}(\Phi(u)-\lambda_n)u_x$ (because of the local behaviour of the K_n). But since this condition is preserved with respect to time evolution it must hold for all time t. Hence, the conditions of theorem 1 are fulfilled. So, the problem whether or not our condition implies asymptotic decomposition into one-solitons is a question whether suitable boundary conditions at infinity are fulfilled for the one-soliton. For the KdV this is the case; for the Burgers equation this is not. It seems noteworthy that in case we have asymptotic decomposition for large time, then the zero set of the solution is just the set of the different asymptotic speeds of the corresponding one-solitons.

Of course, all these statements do not go through for the case of resonance solitons.

Example 3. We consider the resonance one-soliton of the KdV (8) given by

$$0=(\Phi(u)-\lambda)(\Phi(u)-\lambda)u_x=K_2(u)-2\lambda K_1(u)+\lambda^2 u_x \tag{16}$$

where $\Phi(u)$ is the hereditary operator (7). This equation has the usual one-solitons as solutions. We can obtain another solution of this differential equation in the limit $\lambda_1, \lambda_2 \to \lambda$ from the solution of the differential equation

$$K_2(u)-(\lambda_1+\lambda_2)K_1(u)+\lambda_1\lambda_2 u_x=0.$$

One solution of this equation is the two-soliton with asymptotic speeds $\lambda_1=4k_1^2$, $\lambda_2=4k_2^2$ which we can take from the literature [11] as

$$u=2(k_2^2-k_1^2)\frac{k_2^2\operatorname{cosech}^2(\gamma_2)+k_1^2\operatorname{sech}^2(\gamma_1)}{(k_2\coth(\gamma_2)-k_1\tanh(\gamma_1))^2} \tag{17}$$

with

$$\begin{aligned}\gamma_1&=k_1x+4k_1^3t+\delta_1\\ \gamma_2&=k_2x+4k_2^3t+\delta_2\end{aligned} \tag{18}$$

where δ_1, δ_2 are arbitrary constants. For real choices of δ_1, δ_2 the limit $k_1, k_2 \to k$ yields zero. But this changes if we move the δ into the complex plane. For example

$$\delta_1=\delta+\mathrm{i}\tfrac{1}{2}\pi-(k_1-k_2)x_0 \qquad \delta_2=\delta$$

gives for this limit the resonance soliton

$$u(x,t)=16k^2\frac{2\sinh^2(\gamma)-k\Delta\sinh(2\gamma)}{(\sinh(2\gamma)-2k\Delta)^2} \tag{19}$$

where

$$\begin{aligned}\gamma&=kx+4k^3t+\delta\\ \Delta&=(x-x_0)+12k^2t.\end{aligned} \tag{20}$$

Certainly, the time evolution of this solution of the KdV is far from decomposing asymptotically into travelling waves. In the study of the KdV, solutions like this one are sometimes neglected because it has a pole (of second order). Singular solutions like this have been studied for the KdV in [12] and the corresponding non-singular solutions for the sine-Gordon equation in [13]. This example shows clearly that the question whether or not resonance solitons occur is not a principal one but rather a question depending on the special nature of the functions being admitted as solutions.

3. The spectral problem connected to the auto-Bäcklund relation

We first give some heuristic arguments which show that it is not at all surprising that auto-Bäcklund relations lead to isospectral problems for the flows under consideration. We consider the same situation as before and we recall two of the basic heuristic facts about auto-Bäcklund transformations $B(u, \bar{u}, \lambda)=0$.

(i) For suitable choices of the parameter λ the 'implicit function' $B(u, \bar{u}, \lambda)=0$ connects N solitons $\bar{u}$ with $(N+1)$ solitons u, where the 'speed' of the additional soliton is determined by λ (see [9]).

(ii) The Bäcklund transformation is the same [14] for the whole hierarchy of the $K_n(u)$, $n=0,1,2,\ldots$. This means

$$B_u[K_n(u)]+B_{\bar u}[K_n(\bar u)]=0 \tag{21}$$

for $n=0,1,2,\ldots$. Now, recalling that the $K_n(u)$ are generated by $\Phi(u)$, we find

$$B_u[\Phi(u)K(u)]+B_{\bar u}[\Phi(\bar u)K(\bar u)]=0$$

which leads, under the assumption of invertibility of the linear operators B_u and $\bar B_u$, to the well known [14, 15] transformation formula for $\Phi(u)$:

$$\Phi(\bar u)=-B_{\bar u}^{-1}B_u\Phi(u)B_u^{-1}B_{\bar u}. \tag{22}$$

For all known hereditary operators this formula can be checked directly. As a consequence

$$\bar\omega=B_{\bar u}^{-1}B_u[\omega] \tag{23}$$

must be an eigenvector of $\Phi(\bar u)$ whenever ω is one of $\Phi(u)$. Furthermore, the eigenvalues remain invariant under this transformation. Now, consider an $(N+1)$ soliton u and some N soliton $\bar u$ connected to u via $B(u,\bar u,\lambda_N)=0$. For u and $\bar u$ we have the following representations:

$$u_x=\sum_{n=0}^{N}\omega_n \tag{24a}$$

$$\bar u_x=\sum_{n=0}^{N-1}\bar\omega_n \tag{24b}$$

in eigenvectors of $\Phi(u)$ and $\Phi(\bar u)$, respectively. Apply (21) and (23) to (24a) in order to find

$$\bar u_x=\sum_{n=0}^{N-1}\bar\omega_n+B_{\bar u}^{-1}B_u[\omega_N].$$

Comparison with (24b) yields that B_u must have annihilated the ω_N, i.e.

$$B_u(u,\bar u,\lambda_N)[\omega_N]=0 \tag{25a}$$

where $\bar u$ is determined by

$$B(u,\bar u,\lambda_N)=0. \tag{25b}$$

This formula depends on the right choice of the parameter λ_N. It should be remarked that in (25) we can change the λ_N, ω_N into λ_n, ω_n by interchanging the indices in the eigenvector decomposition (24). Our approach shows that given u, there is no suitable $\bar u$ for all λ, with $B(u,\bar u,\lambda_N)=0$ such that the kernel of B_u is non-empty, but only for those λ equal to some λ_n, $n=0,\ldots,N$. Thus we have arrived at an eigenvalue problem.

Spectral problem 1. Given a solution u of (1), find those λ such that there is some non-zero vector field ω and some $\bar u$ with

$$B_u(u,\bar u,\lambda)[\omega]=0 \tag{26a}$$

and

$$B(u,\bar u,\lambda)=0. \tag{26b}$$

Remark 4

(i) This spectral problem is invariant under the flow (1) since variation of u by some suitable scalar multiple of ω corresponds to zero variation of $\bar{u}$.

(ii) In addition our heuristic approach shows that the eigenvectors of this spectral problem coincide with those of $\Phi(u)$. So, if possible, a linearisation of this spectral problem must be equivalent to a spectral problem given by a suitable polynomial of $\Phi(u)$.

In general, it happens that if B represents the 'most simple' auto-Bäcklund relation and $\Phi(u)$ the 'most simple' recursion operator then the spectral problem (26) corresponds to the one given by $\Phi(u)$, whereas an iteration of this ABT leads to a suitable polynomial of $\Phi(u)$. Of course, we cannot give a general theorem concerning this fact without going into a classification of the complexity of the operators Φ and B_u (which would go beyond the scope of this paper).

Some readers may not feel at ease with our structural arguments leading to the connection between the ABT and the recursion operator. Therefore, we go through the construction explicitly in the case of the KdV and the Burgers equation.

Example 4. For the Burgers equation (11) the ABT is given by (12). The variational derivative with respect to u yields

$$B_u = \exp(-D^{-1}(u-\bar{u}))D^{-1} - c. \tag{27a}$$

The spectral problem (26a) gives

$$0 = B_u[\omega] = \exp(-D^{-1}(u-\bar{u}))D^{-1}\omega - c\omega. \tag{27b}$$

In order to obtain the spectral problem in explicit form we have to eliminate $\bar{u}$. From (27b) we obtain

$$\exp(-D^{-1}(u-\bar{u})) = -c\omega/D^{-1}\omega$$

which we insert in (12) to obtain

$$cu + c\omega/(D^{-1}\omega) - \lambda = 0.$$

Multiplication with $D^{-1}\omega$ and finding the derivative with respect to x gives

$$\lambda\omega = c[(uD^{-1}\omega)_x + \omega_x] = c\Phi(u)\omega$$

where $\Phi(u)$ is the recursion operator (10). Hence (27b) is equivalent to ω being an eigenvector of $\Phi(u)$.

Example 5. For the KdV the situation is slightly more complicated. A remark seems in order. For differential operators the spectrum very much depends on the boundary conditions which are required for the solutions. For the KdV one usually considers solutions vanishing rapidly at $x = \pm\infty$. But for this solution manifold (9) is certainly not an ABT since not both u and $\bar{u}$ can vanish at $\pm\infty$. But this incompatability is easily repaired by using the fact that the integration D^{-1} is arbitrary up to a constant. Rewriting then gives the ABT

$$B(u, \bar{u}, \lambda) = u + \bar{u} + \tfrac{1}{2}(D^{-1}(u-\bar{u}))^2 + \lambda D^{-1}(u-\bar{u}) = 0 \tag{28a}$$

which is now compatible with the boundary conditions. The variational derivative of (28a) with respect to u yields

$$B_u = I + (D^{-1}(u-\bar{u}))D^{-1} + \lambda D^{-1} \tag{28b}$$

and the spectral problem (26a) reads as follows:

$$0 = \omega + (D^{-1}(u-\bar{u}))D^{-1}\omega + \lambda D^{-1}\omega. \tag{28c}$$

The abbreviation $D^{-1}\omega = v$ gives

$$D^{-1}(u-\bar{u}) = -(v_x/v+\lambda). \tag{28d}$$

Writing $u+\bar{u}$ as $2u-(u-\bar{u})$ and replacing all terms $u-\bar{u}$ in (28a) by (28d) we obtain

$$2u + (v_x/v+\lambda)_x + \tfrac{1}{2}(v_x/v+\lambda)^2 - \lambda(v_x/v+\lambda) = 0$$

which is certainly a non-linear eigenvalue equation. By multiplication with v^2 we obtain

$$2uv^2 + v_{xx}v - \tfrac{1}{2}v_xv_x = \tfrac{1}{2}\lambda^2v^2. \tag{28e}$$

If this problem can be linearised there must be operators $A(v)$ and $\Psi(u)$ such that $A(v)v = cv^2$ and $A(v)\Psi(u)v$ is equal to the left-hand side of (28e). Comparison of suitable terms yields

$$D^{-1}vD(v_{xx} + 2uv + 2D^{-1}(uv_x)) = \lambda^2 D^{-1}vDv. \tag{28f}$$

Hence $A(v) = D^{-1}vD$ and $\Psi(u) = D^2 + 2u + 2D^{-1}uD$.

Going back to $\omega = v_x$ we see that ω is an eigenvector of the spectral problem if and only if ω is an eigenvector of $\Phi(u)$, which was given in (7). From this computation we also see that the parameter λ in the ABT corresponds to the square root of the corresponding eigenvalue for $\Phi(u)$.

Observe that, in our heuristic approach, we used the fact that the ABT connects multisolitons. This was done in order to show that the spectral problem 1 is equivalent to that given by a suitable polynomial of $\Phi(u)$. Of course, the converse is also true.

Definition 1. Assume that (21) holds. We say the spectral problem 1 is

(i) polynomially equivalent to $\Phi(u)$ if, for any λ, the kernel of $B_u(u, \bar{u}, \lambda)$ is always contained in an eigenspace of $\Phi(u)$, and

(ii) equivalent to $\Phi(u)$ if, for any λ, either of the kernels $B_u(u, \bar{u}, \lambda)$ or $B_{\bar{u}}(u, \bar{u}, \lambda)$ must be empty and if in addition there is a one-to-one map $\lambda \to \varphi(\lambda)$ such that for any u the pair (λ, ω) is a solution of the spectral problem 1 if and only if ω is an eigenvector of $\Phi(u)$ with eigenvalue $\varphi(\lambda)$.

Our examples show that for the Burgers equation the corresponding spectral problem given by the ABT (12) is equivalent to $\Phi(u)$ (given by (10)). The same holds for the KdV provided the solution manifold under consideration is such that $\Phi(u)$ has only positive eigenvalues (which is usually the case since the eigenvalues of $\Phi(u)$ are the squares of the eigenvalues of the Schrödinger operator).

Theorem 2.

(i) Assume that the spectral problem 1 is polynomially equivalent to $\Phi(u)$, then condition (v) implies that u must be a resonance multisoliton.

(ii) Assume that the spectral problem 1 is equivalent to $\Phi(u)$. Then condition (v) with different $\lambda_0, \ldots, \lambda_N$ is equivalent to u being a multisoliton.

Proof. Let $P(\xi)$ be any polynomial in ξ, then (21), together with $K_n(u) = \Phi(u)^n u_x$, implies

$$B_u[P(\Phi(u))u_x] + B_{\bar{u}}[P(\Phi(\bar{u}))\bar{u}_x] = 0. \tag{29}$$

So if $\bar{u}$ is a resonance multisoliton then there must be a polynomial P_1 such that $P_1(\Phi(\bar{u}))\bar{u}_x = 0$. If the spectral problem 1 is polynomially equivalent to Φ then there must be a polynomial $P_2(\Phi(u))$ annihilating the kernel of B_u. This shows that $B(u, \bar{u}, \lambda) = 0$ connects resonance multisolitons, because (29) then implies $B_u[P_1(\Phi(u))u_x] = 0$, hence $P_2(\Phi(u))P_1(\Phi(u))u_x = 0$ (which yields that u is a resonance multisoliton). Now, (i) follows from the fact that 0 is certainly a resonance multisoliton. Hence all the u_n appearing in condition (v) must be resonance multisolitons. In order to prove (ii), assume that u is a multisoliton, i.e. $u_x = \Sigma_{n=0}^{N} \omega_n$ where ω_n are eigenvectors of $\Phi(u)$ with corresponding eigenvalues $\varphi(\lambda)$. Define $\bar{u}$ by $B(u, \bar{u}, \lambda_N) = 0$. Then, since the kernel of B_u is not empty, the kernel of $B_{\bar{u}}$ must be empty and from $B_u[\omega_N] = 0$ and the fact that

$$P(\Phi) = (\Phi(u) - \varphi(\lambda_0)) \ldots (\Phi(u) - \varphi(\lambda_{N-1})) = \sum b_n \Phi^n$$

annihilates $\Sigma_{n=0}^{N-1} \omega_n$ we obtain, via (29), that $P(\Phi(\bar{u}))\bar{u}_x = 0$. Hence, $\bar{u}$ must be a multisoliton in whose spectral decomposition the λ_N does not occur. Repetition of this argument shows that there must be a construction for u according to condition (v) (with different λ). Now assume that condition (v) with different λ holds. Then using definition 1 (ii) and (29) one easily shows that

$$P(\Phi(u)) = (\Phi(u) - \varphi(\lambda_0))(\Phi(u) - \varphi(\lambda_1)) \ldots (\Phi(u) - \varphi(\lambda_N))$$

annihilates u. Since all the zeros of $P(\xi)$ are different u must be a multisoliton.

4. The more complicated situation

For exactly solvable systems the situation is not always as simple as assumed in the introduction. The complications encountered, for example, in the CDGSK equation are as in the following.

Situation 2.

(i) For an evolution equation

$$u_t = K(u) \qquad u = u(x, t) \tag{1}$$

there is a hereditary operator $\Phi_1(u)$ which is a strong symmetry (recursion operator) for (1), i.e. [7, 8]

$$\Phi_1'(u)[K(u)] = K'(u)\Phi_1(u) - \Phi_1(u)K'(u) \tag{30}$$

where $\Phi_1'(u)[\ ]$ and $K'(u)$ denote the variational derivatives.

(ii) Equation (1) is translation invariant (i.e. has u_x as a symmetry generator) but is not of the form $u_t = \Phi(u)^M u_x$.

(iii) There is an auto-Bäcklund relation for (1) which is not (in an obvious way) equivalent to the spectral properties of $\Phi(u)$.

Example 6. The CDGSK

$$u_t = u_{xxxxx} + 30u_{xxx}u + 30u_{xx}u_x + 180u^2u_x \tag{31}$$

has the hereditary [5] operator

$$\Phi(u) = \Theta(u)J(u) \tag{32a}$$

as strong symmetry, where

$$\Theta(u) = D^3 + 3(uD + Du) \tag{32b}$$

$$J(u) = 2D^3 + 18(uD + Du) + 6(D^2uD^{-1} + D^{-1}uD^2) + 6(u^2D^{-1} + D^{-1}u^2). \tag{32c}$$

Obviously (31) cannot be of the form required in the introduction. Equation (31) has the following ABT for (31) (see [4]):

$$(u - \bar{u})_x + \{D^{-1}(u - \bar{u})\}^3 + 3(u + \bar{u})D^{-1}(u - \bar{u}) = \lambda. \tag{33}$$

A tiresome computation shows that, in fact, the spectral problem given by (33) is equivalent to Φ (for the N-soliton case see [1]).

Alas, the essential point in most of the considerations which lead to the different characterisations of multisoliton solutions was that the flow under consideration was generated out of u_x by application of $\Phi(u)$. So there seems to be little hope of transferring our results to situation 2.

However, consider an additional assumption as follows.

Assumption 1. There is some hereditary operator Ψ such that $\Phi = \Pi_1(\Psi)$ and such that (1) is of the form $u_t = \Pi_2(\Psi)u_x$, where $\Pi_1(\Psi)$, $\Pi_2(\Psi)$ are polynomials in Ψ.

Now we can actually transfer almost all the results from situation 1 to situation 2.

Before we go into the details of this statement we would like to discuss whether or not this assumption is a reasonable one. At first, this is very unlikely since in the case of the CDGSK no operator like Ψ has been found and furthermore, some symmetry generators, which should exist as a consequence of assumption 1, are obviously missing in the hierarchy of the system [3, 5]. On the other hand, up to now with respect to hereditary operators we have been spoiled by the KdV and the like, where the corresponding hereditary operators have always been nice polynomials in D, D^{-1} and u. There is no reason why this should always be the case. For example, a hereditary symmetry Ψ could be a polynomial in D, D^{-1}, u and some $\bar{u}$, where $\bar{u}$ is determined by u via a very complicated implicit function. And even for those slightly more complicated Ψ, it could be the case that the third power of Ψ is of the smooth form we are accustomed to. Also, the missing symmetries could be explained in the same way; maybe, every third symmetry is of the complicated implicit form mentioned above whereas the others are not.

In fact, we believe that this is just what happens in case of the CDGSK, and there is very good reason to believe this.

One can prove for completely integrable flows on finite-dimensional manifolds that assumption 1 holds whenever situation 2 occurs. Of course, the conscientious reader still must object that the finite case is not at all characteristic of the infinite-dimensional one. But this objection really does not matter at all since multisoliton manifolds are (if only one space variable occurs) finite dimensional. So, since the CDGSK is completely integrable, all the consequences obtainable from assumption 1 for the description of multisoliton manifolds are justified.

Let us now return to the general problem of characterising multisolitons in our more complicated case. We consider situation 2 under the additional assumption 1.

By $\boldsymbol{K}_1$ we denote the linear hull of the $K_n(u)=\Phi(u)^n u_x$, $n=0,1,\ldots$. Since the vector field $K(u)$ describing the flow (1) is not an element of $\boldsymbol{K}_1$ we have to introduce an additional space of vector fields. Let $\boldsymbol{K}_2$ be the linear hull of the $\bar{K}_n(u)=\Psi(u)^n u_x$, $n=0,1,\ldots$. Now, $\boldsymbol{K}_1\subset\boldsymbol{K}_2$ and $K\in\boldsymbol{K}_2$ (assumption 1). Furthermore $\boldsymbol{K}_2$ is Abelian since Ψ is hereditary [8].

Since the results of lemma 1 and theorem 1 are not really results about flows, but rather results about the space of vector fields generated by polynomials in a fixed operator acting on u_x, it is a simple observation that they hold whenever $(\boldsymbol{K},\Phi)$ is replaced by either $(\boldsymbol{K}_1,\Phi)$ or $(\boldsymbol{K}_2,\Psi)$.

But this is not the real problem. The real problem is that for the definition of notions like multisoliton or resonance multisoliton we have to consider the case where $(\boldsymbol{K},\Phi)$ is replaced by $(\boldsymbol{K}_2,\Phi)$ whereas for the computations only the operator Φ is available because Ψ is unknown to us. Fortunately, our limited knowledge about Ψ is not that important.

Remark 5. The following are equivalent.

(i) There is some $L_1\in\boldsymbol{K}_1$ with $L_1(u)=0$.

(ii) There is some $L_2\in\boldsymbol{K}_2$ with $L_2(u)=0$.

Proof.

(i)$\Rightarrow$(ii) is trivial because of $\boldsymbol{K}_1\subset\boldsymbol{K}_2$.

(ii)$\Rightarrow$(i). Let L_2 be of the form $L_2(u)=P_1(\Psi)u_x$ where p_1 is a polynomial. A simple computation with polynomials shows that there is a polynomial $P_2(\Psi)$ such that $P_1(\Psi)P_2(\Psi)$ can be written as a polynomial $P(\Pi_1(\Psi))$ in $\Pi_1(\Psi)=\Phi$ (assumption 1). Hence $L_1(u)=P(\Phi)u_x=0$ and $L_1\in\boldsymbol{K}_1$.

It is obvious that without further knowledge about the relation between Ψ and Φ we cannot say anything about whether or not non-degeneracy conditions can be transferred from $\boldsymbol{K}_1$ to $\boldsymbol{K}_2$ or vice versa. A consequence of remark 5 is that any of the equivalent conditions (i), (iii) or (vi) completely characterises resonance multisolitons. But even when conditions (ii) or (iv) hold we cannot be sure that we have to deal with a genuine multisoliton because we do not know the exact relation between Φ and Ψ. The same holds for condition (v) since this only yields information about the spectral properties of Φ.

Of course, no harm is done by this uncertainty for concrete computations since the multisolitons are special cases of resonance multisolitons. So whenever we can compute the resonance multisolitons then the multisolitons can be picked out by special choices of the integration constants.

A comparison of the (more heuristic) results of this section with the explicit results [1] about the CDGSK shows complete agreement.

(i) The spectral problem given by the ABT (33) leads to eigenvectors of $\Phi(u)$.

(ii) The resonance multisolitons can be characterised by spectral properties of $\Phi(u)$.

(iii) In general the ABT (or $\Phi(u)$) leads to resonance multisolitons instead of genuine multisolitons.

The last statement needs some further explanation. For simplicity we give the necessary arguments in the case of the general one-soliton found in [1].

Recall that the most general one-soliton found in [1] was the general solution of $u_x = \omega$, ω eigenvector of Φ, i.e. a solution of $(\Phi(u) - \lambda)u_x = 0$. In addition we found that in this case the flow $K(u)$ under consideration did not reduce to a scalar multiple of u_x. This excluded our solution u from being a genuine one-soliton. Still it could be a general multisoliton if the factorisation of $(\Phi - \lambda)$ in linear factors of Ψ has only simple roots. But if it were a genuine multisoliton then different choices of integration parameters should lead to different solitons corresponding to these different roots and this was not the case.

Acknowledgment

One of us (RNA) thanks the University of Paderborn for their hospitality and the Bhabha Atomic Research centre for a grant of leave.

References

[1] Aiyer R N, Fuchssteiner B and Oevel W 1986 *J. Phys. A: Math. Gen.* **19** 3755
[2] Sawada K and Kotera T 1974 *Prog. Theor. Phys.* **51** 1355-67
[3] Caudrey P J, Dodd R K and Gibbon J D 1976 *Proc. R. Soc.* A **351** 407-22
[4] Satsuma J and Kaup D J 1977 *J. Phys. Soc. Japan* **43** 692-7
[5] Fuchssteiner B and Oevel W 1982 *J. Math. Phys.* **23** 358-63
[6] Dodd R K and Gibbon J D 1977 *Proc. R. Soc.* A **358** 287-96
[7] Fuchssteiner B 1979 *Nonlinear Anal. TMA* **3** 849-62
[8] Fuchssteiner B 1981 *Prog. Theor. Phys.* **65** 861-76
[9] Ablowitz M J and Segur H 1981 *Solitons and the Inverse Scattering Transform SIAM Stud. Appl. Math.* (Philadelphia: SIAM)
[10] Fuchssteiner B 1986 *Paderborn preprint*
[11] Lamb G L Jr 1980 *Elements of Soliton Theory* (New York: Wiley)
[12] Jaworski M 1984 *Phys. Lett.* **100A** 321-4
[13] Tsuru H and Wadati M 1984 *J. Phys. Soc. Japan* **53** 2908-21
[14] Fokas A S and Fuchssteiner B 1981 *Nonlinear Anal. TMA* **5** 423-32
[15] Fuchssteiner B and Fokas A S 1981 *Physica* **4D** 47-66